The 3-colorability of planar graphs without cycles of length 4, 6 and 9

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Abstract

In this paper, we prove that planar graphs without cycles of length 4, 6, 9 are 3-colorable.

1 Introduction

The well-known Four Color Theorem states that every planar graph is 4-colorable. On the 3-colorability of planar graphs, a famous theorem owing to Grötzsch [6] states that every planar graph without cycles of length 3 is 3-colorable. Therefore, next sufficient conditions that guarantee 3-colorability of planar graphs should always allow the presence of cycles of length 3. In 1976, Steinberg conjectured that every planar graph without cycles of length 4 and 5 is 3-colorable. Erdös [9] suggested a relaxation of Steinberg's Conjecture: does there exist a constant k such that every planar graph without cycles of length from 4 to k is 3-colorable? Abbott and Zhou [1] proved that such a constant exists and $k \leq 11$. This result was later on improved to $k \leq 9$ by Borodin [2] and, independently, Sanders and Zhao [8], and to $k \leq 7$ by Borodin, Glebov, Raspaud and Salavatipour [3]. Besides, much attention was paid to sufficient conditions that forbid cycles of some other certain length. The results concerning four kinds of forbidden length of cycles were obtained in several different papers and summarized in [7]:

Theorem 1.1. *A planar graph is 3-colorable if it has no cycle of length* 4*,* i*,* j *and* k*, where* $5 \leq i < j < k \leq 9$.

A more general problem than Steinberg's was formulated also in [7]:

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Problem 1.2. *What is $\mathcal{A}$, a set of integers between 5 and 9, such that for $i \in \mathcal{A}$, every planar graph with cycles of length neither 4 nor i is 3-colorable?*

It seems very far to settle Problem 1.2, since no element of such a set $\mathcal{A}$ is found. Therefore, a reasonable way to deal with this problem is to ask following question:

Problem 1.3. *What is $\mathcal{B}$, a set of pairs of integers (i, j) with $5 \leq i < j \leq 9$, such that planar graphs without cycles of length 4, i and j are 3-colorable?*

The first step towards Problem 1.3 was made by Xu [11], who proved that a planar graph is 3-colorable if it has neither 5- and 7-cycles nor adjacent 3-cycles. Unfortunately, there is a gap in his proof, as pointed out by Borodin etc. [4], who later on gave a new proof of the same statement. Afterwards, Xu [12] fixed this gap. Hence $(5, 7) \in \mathcal{B}$. Other known elements of $\mathcal{B}$ includes pair (6,8) given by Wang and Chen [10], pair (7,9) given by Lu etc. [7], and pair (6,7) given by Borodin, Glebov and Raspaud [5]. Actually, the theorem proved in [5] states that planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable, which implies $(6, 7) \in \mathcal{B}$.

In this paper, we show that $(6, 9) \in \mathcal{B}$, that is, we prove the following theorem:

Theorem 1.4. *Every planar graph without cycles of length 4, 6, 9 is 3-colorable.*

The graphs considered in this paper are finite and simple. Let G be a plane graph and C a cycle of G. By $Int(C)$ (or $Ext(C)$) we denote the subgraph of G induced by the vertices lying inside (or outside) of C. Cycle C is *separating* if both $Int(C)$ and $Ext(C)$ are not empty. By $\overline{Int}(C)$ (or $\overline{Ext}(C)$) we denote the subgraph of G consisting of C and its interior (or exterior).

Denote by $G[S]$ the subgraph of G induced by S, where either $S \subseteq V(G)$ or $S \subseteq E(G)$. A vertex is a *neighbor* of another vertex if they are adjacent. A *chord* of C is an edge of $\overline{Int}(C)$ that connects two nonconsecutive vertices on C. If $Int(C)$ has a vertex v with three neighbors v_1, v_2, v_3 on C, then $G[\{vv_1, vv_2, vv_3\}]$ is called a *claw* of C. If $Int(C)$ has two adjacent vertices u and v such that u has two neighbors u_1, u_2 on C and v has two neighbors v_1, v_2 on C, then $G[\{uv, uu_1, uu_2, vv_1, vv_2\}]$ is called a *biclaw* of C. If $Int(C)$ has three pairwise adjacent vertices u, v, w such that u, v and w have a neighbor u', v' and w' on C respectively, then $G[\{uv, vw, uw, uu', vv', ww'\}]$ is called a *triclaw* of C (see Figure 1).

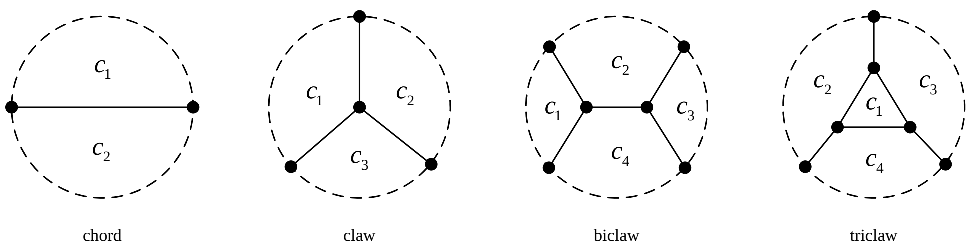

Figure 1: chord, claw, biclaw and triclaw of a cycle

Let C be a cycle and T be one of the chords, claws, biclaws and triclaws of C. We call the graph consisting of C and T a *bad partition* H of C. The boundary of any one of the parts, into which C is divided by H, is called a *cell* of H. Clearly, every cell is a cycle. In case of confusion, let us always order the cells $c_1, \cdots, c_t$ of H in the way as shown in Figure 1. For every cell c_i of H, let k_i be the length of c_i. Then T is further called a (k_1, k_2)-chord, a (k_1, k_2, k_3)-claw, a (k_1, k_2, k_3, k_4)-biclaw or a (k_1, k_2, k_3, k_4)-triclaw, respectively.

Let k be a positive integer. A k-cycle is a cycle of length k. A k^--cycle (or k^+-cycle) is a cycle of length at least (or at most) k. A *good cycle* is a 12^--cycle that has none of claws, biclaws and triclaws. A *bad cycle* is a 12^--cycle that is not good. We say a 9-cycle is *special* if it has a (3,8)-chord or a (5,5,5)-claw.

Let $\mathcal{G}$ be the class of connected plane graphs with neither 4- and 6-cycle nor special 9-cycle.

Instead of Theorem 1.4, it is easier for us to prove the following stronger one:

Theorem 1.5. *Let $G \in \mathcal{G}$. We have*

(1) G is 3-colorable; and

(2) If D, the boundary of the exterior face of G, is a good cycle, then every proper 3-coloring of $G[V(D)]$ can be extended to a proper 3-coloring of G.

This section is concluded with some notations that are used in the next section. Let G be a plane graph. Denote by $d(v)$ the degree of a vertex v, by $|C|$ the length of a cycle C and by $|f|$ the size of a face f. Let k be a positive integer. A *k-vertex* is a vertex of degree k, and a *k-face* is a face of size k. A *k^+-vertex* (or *k^--vertex*) is a vertex of degree at least (or at most) k, and a *k^+-face* (or *k^--face*) is a face of size at least (or at most) k. A *k-path* is a path that contains k edges. A k-cycle containing vertices $v_1, \ldots, v_k$ in cyclic order is denoted by $[v_1 \ldots v_k]$. Denote by $N(v)$ the set of neighbors of a vertex v. Let $N_H(v) = N(v) \cap V(H)$ whenever v is a vertex of a subgraph H of G. A vertex is *external* if it lies on the exterior face, *internal* otherwise. A vertex incident with a triangle is called a *triangular vertex*. We say a vertex is *bad* if it is an internal triangular 3-vertex; *good* otherwise. A path is a *splitting path* of a cycle C if it has two end-vertices on C and all other vertices inside C. We say a path is *good* if it contains only internal 3-vertices and has an end-edge incident with a triangle. A cycle or a face C is *triangular* if C is adjacent to a triangle T. Furthermore, if C is a cycle and $T \in \overline{Ext}(C)$, then we say C is an *ext-triangular* cycle. A triangular 7-face is *light* if it has no external vertex and every incident nontriangular vertex has degree 3.

2 Proof of Theorem 1.5

Suppose to the contrary that Theorem 1.5 is false. From now on, let G be a counterexample to Theorem 1.5 with fewest vertices. Actually, G violates the second conclusion of Theorem 1.5,

since conclusion (2) implies conclusion (1). We still use D to denote the boundary of the exterior face of G, and let ϕ be a proper 3-coloring of $G[V(D)]$ which cannot be extended to a proper 3-coloring of G. Clearly, D is a good cycle. By the minimality of G, D has no chord.

2.1 Structural properties of minimal counterexample G

Lemma 2.1. *Every internal vertex of G has degree at least 3.*

Proof. Suppose to the contrary that G has an internal vertex v such that $d(v) \leq 2$. We can extend ϕ to $G - v$ by the minimality of G, and then to G by coloring v different from its neighbors. □

Lemma 2.2. *G is 2-connected and therefore, the boundary of each face of G is a cycle.*

Proof. Otherwise, we may assume that G has a pendant block B with cut vertex v such that $B - v$ does not intersect with D. We first extend ϕ to $G - (B - v)$, and then 3-color B such that the color assigned to v is unchanged. □

Lemma 2.3. *G has no separating good cycle.*

Proof. Suppose to the contrary that G has a separating good cycle C. We extend ϕ to $G - Int(C)$. Furthermore, since C is a good cycle, the color of C can be extended to its interior. □

One can easily conclude following three lemmas.

Lemma 2.4. *Every 9^--cycle of G is facial except that an 8-cycle of G might have a chord, which is a (3,7)- or (5,5)-chord.*

Lemma 2.5. *Let $H \in \mathcal{G}$. If C is a bad cycle of H, then C has length either 11 or 12. Furthermore, if $|C| = 11$, then C has a (3,7,7)- or (5,5,7)-claw; if $|C| = 12$, then C has a (5,5,8)-claw, a (3,7,5,7)- or (5,5,5,7)-biclaw, or a (3,7,7,7)-triclaw.*

Lemma 2.6. *Every bad cycle C of G is adjacent to at most one triangle. Furthermore, if C is ext-triangular, then C has either a (5,5,7)-claw or a (5,5,5,7)-biclaw.*

Lemma 2.7. *Let P be a splitting path of D which divides D into two cycles D' and D''.*

(1) If $|P| = 2$, then there is a 3-face between D' and D'';

(2) If $|P| = 3$, then there is a 5-face between D' and D'';

(3) If $|P| = 4$, then there is a 5- or 7-face between D' and D'';

(4) If $|P| = 5$, then there is a 9^--cycle between D' and D''.

Proof. Since D has length at most 12, we have $|D'| + |D''| = |D| + 2|P| \leq 12 + 2|P|$. Recall that every 7^--cycle of G is a facial cycle by Lemma 2.4.

(1) Let $P = xyz$. Suppose to the contrary that $|D'|, |D''| \geq 5$. By Lemma 2.1, y has a neighbor other than x and z, say y'. It follows that y' is internal since otherwise D is a bad cycle with a claw. Without loss of generality, let y' lie inside D'. Thus $|D'| \geq 11$ by Lemma 2.3. Since $|D'| + |D''| \leq 16$, we have $|D'| = 11$ and $|D''| = 5$. Now D' has a claw by Lemma 2.5, which implies that D has a biclaw, a contradiction.

(2) Let $P = wxyz$. Suppose to the contrary that $|D'|, |D''| \geq 7$. Let x' and y' be a neighbor of x and y not on P, respectively. If both x' and y' are external, then D has a biclaw. Hence, we may assume x' lies inside D'. By Lemma 2.5 and inequality $|D'| + |D''| \leq 18$, we have $|D'| = 11$ and $|D''|$=7. Thus D' has a claw which divides D' into three faces. Since D'' is facial, y' can only coincide with x'. Now D has a triclaw.

(3) Let $P = vwxyz$. Suppose to the contrary that $|D'|, |D''| \geq 8$. Since $|D'| + |D''| \leq 20$, we have $|D'|, |D''| \leq 12$. If G has an edge e connecting two nonconsecutive vertices on P, then e together with P can form only a triangle. Without loss of generality, let $e = wy$ and e belongs to $\overline{Int}(D')$. Now path $vwyz$ is a splitting 3-path of D and hence D' is a 6-cycle with a (3,5)-chord, a contradiction. Therefore, no pair of nonconsecutive vertices on P are adjacent.

Let w', x', y' be a neighbor of w, x, y not on P, respectively. If x' is external, say xx' is a chord of D', then both of paths $vwxx'$ and $x'xyz$ are splitting 3-paths of D. It follows that D' is an 8-cycle with a (5,5)-chord xx'. Hence y' has no other possibility but to lie inside of D'', and so does w'. By noticing that w' cannot coincide with y', we know D'' is a bad 12-cycle. It follows that G has an edge connecting w' and y', which yields a special 9-cycle of G. Therefore, vertex x' is internal.

We may assume x' lies inside of D'. Thus D' is a bad 11- or 12-cycle, which implies D'' has length 8 or 9. If $|D''| = 9$, then D'' is facial and D' is a bad 11-cycle with a claw, which is impossible because of the locations of w' and y'. Hence we may assume $|D''| = 8$. It follows that not both w' and y' lie in $\overline{Int}(D'')$ and that w', x', y' are pairwise distinct. Now G has a 4-cycle that is either $[wxx'w']$ or $[xyy'x']$, a contradiction.

(4) Let $P = uvwxyz$. Suppose to the contrary that $|D'|, |D''| \geq 10$. Since $|D'| + |D''| \leq 22$, we have $|D'|, |D''| \leq 12$. By similar argument as in (3), one can conclude that G has no edge connecting two nonconsecutive vertices on P. Let v', w', x', y' be a neighbor of v, w, x, y not on P, respectively.

We claim that both vertices w' and x' are internal. Otherwise, let ww' be a chord of D'. Since both $uvww'$ and $w'wxyz$ are splitting paths of D, D' is a 10-cycle with a (5,7)-chord ww'. Thus all of v', x' and y' belong to $\overline{Int}(D'')$. If x' is external, then similarly, D'' is a 10-cycle with a (5,7)-chord xx', which is impossible because of the location of y'. Hence, we may assume that x' lies inside D''. Furthermore, v' also lies inside D'', since otherwise G has 3-face $[uvv']$ adjacent to a 5-face. Clearly, $v' \neq x'$. Hence D'' is a bad 12-cycle containing two adjacent vertices v' and x' inside. A contradiction is obtained by noticing both the location of y' and the specific interior of

D''.

Let w' lie inside D'. If x' lies inside D'', then both D' and D'' are bad 11-cycles. It follows that $v' = w', y' = x'$, and both D' and D'' have a $(3, 7, 7)$-claw, yielding special 9-cycles of G. Hence, we may assume that x' lies also inside D'. It follows that x' coincide with w', since otherwise the adjacency of x' and w' gives a 4-cycle of G. Thus D' is a bad cycle with either a $(3, 7, 7)$-claw or a (3,7,5,7)-biclaw, which implies both v' and y' belong to $\overline{Int}(D'')$. If v' lies on D'', then G has triangle $[uvv']$ adjacent to an 8-cycle of D', a contradiction. Hence we may assume v' lies inside D'' and so does y'. It follows that either $v' = y'$ or $v'y' \in E(G)$, yielding 6-cycles of G in both cases.

The proof of this lemma is completed. □

Lemma 2.8. *Let G' be a plane graph obtained from G by a graph operation T.*

Let T consist of deleting a nonempty set of internal vertices and either identifying two vertices or adding an edge between two nonadjacent vertices. If after T we

(a) identify no two vertices on D, and create no edge connecting two vertices on D, and

(b) create neither 6^--cycle nor ext-triangular 7- or 8-cycle,

then ϕ can be extended to G'.

Let T consist of deleting a nonempty set S of internal vertices and identifying two edges u_1u_2 and v_1v_2 so that u_1 is identified with v_1. For $i \in \{1, 2\}$, let T_i denote the operation on G that consists of deleting all vertices in S and identifying u_i and v_i. If at least one of u_1u_2 and v_1v_2 is contained in no 8^--cycle of $G - S$, and if conditions (a) and (b) above hold for both T_1 and T_2, then ϕ can be extended to G'.

Proof. First let T consist of deleting a nonempty set of internal vertices and identifying two other vertices t_1 and t_2. Let t' denote the vertex obtained from t_1 and t_2 after T. Conditions (a) and (b) implies (i) to show $G' \in \mathcal{G}$, it suffices to show G' has no special 9-cycles; and (ii) D bounds G' and ϕ is a proper 3-coloring of $G'[V(D)]$. Therefore, ϕ can be extended to G' by the minimality of G if we can show both that G' has no special 9-cycles and that D is good in G'.

Suppose G' has a special 9-cycle C. Let H be a bad partition of C. We have $t' \in V(H)$ since otherwise C is a special 9-cycle in G. Condition (b) implies that every vertex of $N_H(t')$ is adjacent to precisely one of t_1 and t_2 in G. If all the vertices of $N_H(t')$ is adjacent to t_1, then C is a special 9-cycle in G. Hence, we may assume that $N_H(t')$ has a vertex adjacent to t_2 and similarly, has another vertex adjacent to t_1. Thus after T a cell of H containing t' is created, that is, we have created a 3- or 5-cycle or an ext-triangular 8-cycle, contradicting (b). Therefore, G' has no special 9-cycle.

Suppose D is bad in G'. Let H be a bad partition of D. We have $t' \in V(H)$ since otherwise D is bad in G. If t' has degree 2 in H, then $t_1, t_2 \in V(D)$ since otherwise D is bad in G. Now we identify two vertices on D, contradicting (a). Hence t' has degree 3 in H. Similarly as paragraph above, we may assume that $N_H(t')$ has a vertex w_1 adjacent to t_1 and two other vertices w'_2, w''_2

adjacent to t_2 in G. It follows that H has two cells containing either $w_1t'w_2'$ or $w_1t'w_2''$ created by T. Clearly, $G' \in \mathcal{G}$. Hence, after T we create a 3- or 5-cycle, or an ext-triangular 7-cycle, contradicting (b). Therefore, D is good in G'.

Next let T consist of deleting a nonempty set of internal vertices and adding an edge e between two nonadjacent vertices. Similarly, to complete the proof in this case, it suffices to guarantee that G' has no special 9-cycles and that D is good in G'.

Suppose G' has a special 9-cycle C. Let H be a bad partition of C. We have $e \in E(H)$ since otherwise C is a special 9-cycle of G. Hence, every cell of H containing e is created, which implies that we have created a 3- or 5-cycle or an ext-triangular 8-cycle, contradicting (b).

Suppose D is bad in G'. Let H be a bad partition of $Int(D)$. Similarly, one can conclude that every cell of H containing e is created. Since $e \notin E(D)$ and $G' \in \mathcal{G}$, we create a 3- or 5-cycle or an ext-triangular 7-cycle, a contradiction.

At last, let T consist of deleting all vertices in S and identifying two edges u_1u_2 and v_1v_2. Denote by w_1 the vertex of G' obtained from u_1 and v_1 after T, and by w_2 one obtained from u_2 and v_2. Since condition (a) holds for both T_1 and T_2, D bounds G' and ϕ is a proper 3-coloring of $G'[V(D)]$.

Suppose we create a 6^--cycle C' after T. Since condition (b) holds for both T_1 and T_2, we have $w_1, w_2 \in V(C')$ and furthermore, one of the two paths of C' between w_1 and w_2 connects u_1 and u_2, and the other connects v_1 and v_2. Clearly, w_1 and w_2 are nonconsecutive on C', since otherwise C' is a 6^--cycle of G. It follows that both u_1u_2 and v_1v_2 are contained in a 5^--cycles of $G - S$, a contradiction. Therefore, we create no 6^--cycle by T. Furthermore, by a similar argument, one can conclude that we create no ext-triangular 7- or 8-cycle by T.

Suppose we create a special 9-cycle C after T. Let H be a bad partition of C. Clearly, no cell of H is created by T. It follows that G has a 2-path between u_1 and u_2 and a 7-path between v_1 and v_2 so that edge w_1w_2 is a (3,8)-chord of C, since otherwise C is a special 9-cycle of G. Now both u_1u_2 and v_1v_2 are contained in an 8^--cycle of G, a contradiction. Therefore, we create no special 9-cycle after T.

Suppose D is bad in G'. Let H be a bad partition of D. Notice that by T we identify one pair of edges, and that each cell of H has more than one edge shared with some other cell. If no cell of H is created by T, then D is bad in G. Hence, we may assume that H has a cell C_H that is created by T. Recall that condition (b) holds for T, too. It follows that H has either a (5,5,7)- or (5,5,8)-claw or a (5,5,5,7)-biclaw, and C_H is the cell of length at least 7. Furthermore, since D is unchanged and no 6^--cycle is created after T, it is impossible that we create C_H but no other cells of H by T. Therefore, D is good in G'.

By the conclusions above, ϕ can be extended to G' because of the minimality of G. □

Lemma 2.9. *Every face of G contains no good path.*

Proof. Suppose to the contrary that G has a k-face f that contains a good path Q. Since $G \in \mathcal{G}$,

we have $k \geq 7$. Let $f = [v_1 \dots v_k]$ and $Q = v_2 \dots v_5$. Let t be a common neighbor of v_2 and v_3 not on Q, and x be a neighbor of v_4 other than v_3 and v_5. Clearly, $x \neq v_1$. We do a graph operation T on G as follows: delete all vertices on Q and identify v_1 and x, obtaining a plane graph G'.

Suppose that through T we identify two vertices on D, or create an edge connecting two vertices on D. G has a splitting 4- or 5-path P of D that contains path $v_1 \dots v_4 x$. Thus by Lemma 2.7, G has a 9^--cycle C formed by P and D. Clearly, C is a good cycle and thus none of t and v_5 lies inside C, which implies t lies on C. Now C has two chords tv_2 and tv_3, a contradiction with Lemma 2.4. Therefore, item (a) in Lemma 2.8 holds for T.

Suppose that through T we create a 6^--cycle or an ext-triangular 7- or 8-cycle. Thus $G - v_5$ has a 12^--cycle C containing path $v_1 \dots v_4 x$, such that $\overline{Ext}(C)$ has a triangle adjacent to C with common edge on $C - \{v_2, v_3, v_4\}$ when $|C| \in \{11, 12\}$. It follows that $t \notin V(C)$ since otherwise G has a 6^--face adjacent to triangle $[tv_2v_3]$. Hence, C is a bad cycle containing either t or v_5 inside. Now C is adjacent to two triangles, contradicting Lemma 2.6. Therefore, item (b) in Lemma 2.8 holds for T.

Hence ϕ can be extended to G' by Lemma 2.8. Next we extend ϕ from G' to G: first properly color v_5 and v_4 in turn, then v_2 and v_3 can be properly colored since v_1 and v_4 receive different colors. $\square$

Lemma 2.10. *G has no k-face containing k internal 3-vertex, where $k \in \{5, 7\}$.*

Proof. Suppose to the contrary that G has such a k-face f. Let $f = [v_1 \dots v_k]$. For $1 \leq i \leq k$, denote by u_i the neighbor of v_i not on f. Clearly, vertices $u_1, \cdots, u_k$ are pairwise distinct.

(1) Let $k = 5$. Since G has no special 9-cycles, f has a vertex incident with two 7^+-faces. Without loss of generality, let v_1 be such a vertex. We do a graph operation T on G as follows: delete all the vertices on f and insert an edge between u_5 and u_2. Denote by G' the resulting plane graph.

Suppose that $u_2, u_5 \in V(D)$. As a splitting 4-path of D, path $u_5v_5v_1v_2u_2$ together with D forms a 5- or 7-face of G, an obvious contradiction. Therefore, item (a) holds for T.

Suppose through T we create a 6^--cycle or an ext-triangular 7- or 8-cycle. Then $G - \{v_3, v_4\}$ has a 11^--cycle C containing path $u_5v_5v_1v_2u_2$ such that $\overline{Ext}(C)$ has a triangle adjacent to C with common edge on $C - \{v_5, v_1, v_2\}$ when $|C| \in \{10, 11\}$. If C is a good cycle, then none of u_1, v_3 and v_4 lies inside C, which implies $u_1 \in V(C)$. Now u_1v_1 divides C into two cycles C_1 and C_2. On one hand, since v_1 is incident with two 7^+-faces, we have $|C_1|, |C_2| \geq 7$. On the other hand, we have $|C_1| + |C_2| = |C| + 2 \leq 13$. An contradiction is obtained. Hence, we may assume C is a bad 11-cycle. It follows that C has a $(5, 5, 7)$-claw by Lemma 2.6, which is impossible since now either C contains two vertices v_3 and v_4 inside or $\overline{Int}(C)$ has two 7^+-faces. Therefore, item (b) holds for T.

Hence by Lemma 2.8, ϕ can be extended to G'. Notice that u_1 receives a color different from at least one of u_2 and u_5. Without loss of generality, let us say u_2. We extend ϕ from G' to G in

following way: color v_2 same as u_1, then v_3, v_4, v_5 and v_1 can be properly colored in turn.

(2) Let $k = 7$. We do following operation T on G: delete all vertices on f and insert an edge between u_1 and u_5, obtaining a plane graph G'.

Suppose both u_1 and u_5 belong to D. Let $P = u_1v_1v_7v_6v_5u_5$. Since P is a splitting path of D, G has a 9^--cycle C formed by P and D by Lemma 2.7. Clearly, C is good. Thus $u_6, u_7 \in V(C)$. Now C has two chords, a contradiction with Lemma 2.4. Therefore, item (a) holds for T.

Suppose through T we create a 6^--cycle or an ext-triangular 7- or 8-cycle. Then $G-\{v_2, v_3, v_4\}$ has a 12^--cycle C containing path P such that $\overline{Ext}(C)$ has a triangle adjacent to C with common edge on $C - \{v_1, v_7, v_6, v_5\}$ when $|C| \in \{11, 12\}$. If C is a good cycle, then both u_6 and u_7 lie on C. Since $|C| \leq 12$, edges v_6u_6 and v_7u_7 divide C into three cycles, each of which has length 5. It follows that $|C| = 11$ and hence $\overline{Int}(C)$ has a 5-face adjacent to a triangle, a contradiction. Hence, we may assume C is a bad cycle. By Lemma 2.6, C has either a $(5, 5, 7)$-claw or a $(5, 5, 5, 7)$-biclaw, which is impossible obviously. Therefore, item (b) holds for T.

Hence by Lemma 2.8, ϕ can be extended to G'. Furthermore, ϕ can be extended from G' to G in a similar way as part (1) of this lemma. □

Lemma 2.11. *G has no two 7-faces $[xv_1 \dots v_6]$ and $[xu_1 \dots u_6]$ such that x is their unique common vertex, u_1 and v_1 are adjacent, both x and u_1 are internal 4-vertices, and all other vertices on these two 7-faces are internal 3-vertices.*

Proof. Suppose to the contrary G has such two 7-faces. Let $f = [xv_1 \dots v_6]$ and $g = [xu_1 \dots u_6]$. Let y and z be the neighbors of u_1 and v_6 not on $f \cup g$, respectively. We do the following operation T on G: delete both $V(f)$ and $V(g)$, and identify z and y, obtaining a plane graph G'.

Suppose through T we identify two vertices on D, or create an edge connecting two vertices on D. Then G has a splitting 4- or 5-path P of D containing path yu_1xv_6z. It follows from Lemma 2.7 that G has a 9^--cycle C formed by P and D. Hence, C is a good cycle and thus not separating, contradicting that C has either u_2 or v_1 inside. Therefore, item (a) holds for T.

Suppose through T we create a 6^--cycle or an ext-triangular 7- or 8-cycle. Then $G-V(f)\cup V(g)$ has a 8^--path between y and z, which together with path yu_1xv_6z form a 12^--cycle C. It follows that G has at most three vertices inside C, contradicting the fact that now either $u_2, \dots, u_6$ or $v_1, \dots, v_5$ lie inside C. Therefore, item (b) holds for T.

Hence by Lemma 2.8, ϕ can be extended to G'. We further extend ϕ from G' to G in following way: first color x same as y, then $u_6, \dots, u_1$ can be properly colored in turn, and so do $v_1, \dots, v_6$. □

Lemma 2.12. *G has no 8-cycle $[xyzu_1 \dots u_5]$ with a chord xz such that z is an internal 4-vertex and all other vertices of this 8-cycle are internal 3-vertices.*

Proof. Suppose to the contrary that G has such an 8-cycle C. Let z' and y' be the neighbors of z and y not on C, respectively. We remove C from G to obtain a plane graph G' with fewer

vertices. By the minimality of G, ϕ can be extended to G'. We complete the proof by extending ϕ from G' to G in following way: if z' and y' receive a same color, then we color x same as z' and finally, $u_5, \ldots, u_1, z, y$ can be properly colored in turn; otherwise, we color z same as y', and then $u_1, \ldots, u_5, x, y$ can be properly colored in turn. □

Lemma 2.13. *G has no 9-face $[u_1 \ldots u_9]$ such that $u_1, u_2, u_3, u_5, u_6, u_7$ are six bad vertices and u_4 is a 4-vertex incident with two 3-faces.*

Proof. Suppose to the contrary G has such a 9-face f. G has four 3-faces $[xu_1u_2]$, $[yu_3u_4]$, $[zu_4u_5]$, $[wu_6u_7]$ adjacent to f. Let $S = \{u_1, u_2, u_3, u_5, u_6, u_7\}$. We apply following graph operation T on G to obtain a plane graph G' with fewer vertices: delete all vertices of S and identify two edges u_8u_9 and zu_4 so that u_8 is identified with z. Denote by T_1 (or T_2) the graph operation on G that consists of deleting all vertices in S and identifying u_8 and z (or u_9 and u_4). Similarly as the proof of Lemma 2.9, one can conclude that items (a) and (b) hold for both T_1 and T_2. Besides, u_4z is contained in no 8^--cycle of $G - S$. Hence, ϕ can be extended to G' by Lemma 2.8. Furthermore, we can extend ϕ from G' to G in a similar way as Lemma 2.9.

□

2.2 Discharging in G

Let $V = V(G)$, $E = E(G)$, and F be the set of faces of G. Denote by f_0 the exterior face of G. Give initial charge $ch(x)$ to each element x of $V \cup F$, where $ch(f_0) = d(f_0) + 4$, $ch(v) = d(v) - 4$ for $v \in V$, and $ch(f) = |f| - 4$ for $f \in F \setminus \{f_0\}$. Discharge the elements of $V \cup F$ according to the following rules:

$R1$. Every 3-face receives $\frac{1}{3}$ from each incident vertex.

$R2$. Let v be an internal 3-vertex and f be a face containing v.

(1) Vertex v receives $\frac{1}{4}$ from f if $d(f) = 5$.

(2) Suppose $d(f) \geq 7$. Let a and b denote the lengths of two faces containing v other than f, and $a \leq b$. Vertex v receives from f charge $\frac{2}{3}$ if $a = 3$, charge $\frac{1}{2}$ if $a = b = 5$, charge $\frac{3}{8}$ if $a = 5$ and $b \geq 7$, and charge $\frac{1}{3}$ if $a \geq 7$.

$R3$. Let v be an internal 4-vertex and f be a 7^+-face containing v.

(1) If v is incident with precisely two 3-faces, then v receives $\frac{1}{3}$ from f.

(2) If v is incident with precisely one 3-face that is adjacent to f, then v receives $\frac{1}{6}$ from f.

$R4$. Let f be a light 7-face adjacent to a 3-face T on edge xy, z be the vertex on T other than x and y, and h be the face containing edge yz other than T.

(1) If $d(x) = 3$ and $d(y) \geq 5$, then y sends $\frac{1}{24}$ to f.

(2) If $z \in V(D)$, then z sends $\frac{5}{24}$ to f through T.

(3) If $d(x) = 3, d(y) = 4, z \notin V(D)$ and $d(z) \geq 4$, then h sends $\frac{5}{24}$ to f through y.

$R5$. The exterior face f_0 sends $\frac{4}{3}$ to each incident vertex.

$R6$. Let v be an external vertex and f be a 5^+-face containing v other than f_0.

(1) If $d(v) = 2$, then v receives $\frac{2}{3}$ from f.

(2) Suppose $d(v) = 3$. If v is triangular, then v receives $\frac{1}{12}$ from f; otherwise, v sends $\frac{1}{12}$ to f.

(3) If $d(v) \geq 4$, then v sends $\frac{1}{3}$ to f.

Let $ch^*(x)$ denote the final charge of each element x of $V \cup F$ after discharging. On one hand, by Euler's formula we deduce $\sum\limits_{x \in V \cup F} ch(x) = 0$. Since the sum of charge over all elements of $V \cup F$ is unchanged, we have $\sum\limits_{x \in V \cup F} ch^*(x) = 0$. On the other hand, we show that $ch^*(x) \geq 0$ for $x \in V \cup F$ and $ch^*(x_0) > 0$ for some vertex x_0. Hence, this obvious contradiction completes the proof of Theorem 1.5.

It remains to show that $ch^*(x) \geq 0$ for $x \in V \cup F$ and $ch^*(x_0) > 0$ for some vertex x_0.

Claim 2.14. *$ch^*(f) \geq 0$ for $f \in F$.*

Denote by $V(f)$ the set of vertices of f.

First suppose that f contains no external vertex.

Let $|f| = 3$. By $R1$, we have $ch^*(f) = |f| - 4 + 3 \times \frac{1}{3} = 0$.

Let $|f| = 5$. Lemma 2.10 implies that f contains at most four 3-vertices. Hence, we have $ch^*(f) \geq |f| - 4 - 4 \times \frac{1}{4} = 0$ by $R2(1)$.

Let $|f| = 7$. If G has no 3-face adjacent to f, then f sends at most $\frac{1}{2}$ to each incident 3-vertex by $R2(2)$. Since Lemma 2.10 implies that f contains at most six 3-vertices, we have $ch^*(f) \geq |f| - 4 - 6 \times \frac{1}{2} = 0$. Hence, we may assume that f is adjacent to a 3-face $T = [xyz]$ on edge xy, where $d(x) \leq d(y)$. Since G has no special 9-cycle, f is adjacent to no other 3-face than T. Notice that now only rules $R2(2), R3(2)$ and $R4(3)$ might make f send charge out.

Suppose $d(y) = 3$. In this case f sends $\frac{2}{3}$ to both x and y, and at most $\frac{1}{2}$ to each of other incident 3-vertices. Moreover, it follows from Lemma 2.9 that f contains at least two 4^+-vertices. Hence, we have $ch^*(f) \geq |f| - 4 - 2 \times \frac{2}{3} - 3 \times \frac{1}{2} > 0$.

Suppose $d(x) = 3$ and $d(y) = 4$. In this case f sends $\frac{2}{3}$ to x, at most $\frac{1}{6}$ to y, and at most $\frac{3}{8}$ to the neighbor of x on f other than y. If z is not an internal 3-vertex, then f receives charge $\frac{5}{24}$ either from z by $R6(3)$ or from the face containing yz other than T by $R4(3)$, yielding $ch^*(f) \geq |f| - 4 - \frac{2}{3} - \frac{1}{6} - \frac{3}{8} - 4 \times \frac{1}{2} + \frac{5}{24} = 0$. Hence, we may assume z is an internal 3-vertex. Since Lemma 2.12 implies f is not light, we have $ch^*(f) \geq |f| - 4 - \frac{2}{3} - \frac{1}{6} - 4 \times \frac{1}{2} > 0$.

It remains to suppose $d(x) \geq 4$. In this case, f might send charge out through x and y by $R4(3)$. If f is not light, then $ch^*(f) \geq |f| - 4 - 2(\frac{1}{6} + \frac{5}{24}) - 4 \times \frac{1}{2} > 0$. If $d(y) \geq 5$, then f sends

nothing to y or through y, yielding $ch^*(f) \geq |f| - 4 - (\frac{1}{6} + \frac{5}{24}) - 5 \times \frac{1}{2} > 0$. Hence, we may assume that f is light and $d(x) = d(y) = 4$. Lemma 2.11 implies that f sends nothing out through x or y. It follows that $ch^*(f) \geq 7 - 4 - 2 \times \frac{1}{6} - 5 \times \frac{1}{2} > 0$.

Let $|f| = 8$. Since f sends at most $\frac{1}{2}$ to each incident vertex by $R2(2)$, we have $ch^*(f) \geq 8 - 4 - 8 \times \frac{1}{2} = 0$.

Let $|f| \geq 9$. We define

$$A(f) = \{v\colon\ \textit{uvw is a path on f, both u and w are bad, and v is good}\},$$

$$B(f) = \{v\colon\ \textit{uvw is a path on f, u is bad, and both v and w are good}\},$$

$$C(f) = \{v\colon\ \textit{uvw is a path on f, and all of u, v and w are good}\},$$

$$D(f) = \{v\colon\ \textit{v is a bad vertex on f}\}.$$

Clearly, $A(f), B(f), C(f)$ and $D(f)$ are pairwise disjoint sets whose union is $V(f)$. By our rules, f sends at most $\frac{1}{3}$ to each vertex in $A(f)$, at most $\frac{3}{8}$ in total to and through each vertex in $B(f)$, at most $\frac{1}{2}$ in total to and through each vertex in $C(f)$ and $\frac{2}{3}$ to each vertex in $D(f)$. Hence, we have

$$\begin{aligned} ch^*(f) &\geq |f| - 4 - \frac{1}{3}|A(f)| - \frac{3}{8}|B(f)| - \frac{1}{2}|C(f)| - \frac{2}{3}(|f| - |A(f)| - |B(f)| - |C(f)|) \\ &= \frac{1}{3}|A(f)| + \frac{7}{24}|B(f)| + \frac{1}{6}|C(f)| + \frac{1}{3}|f| - 4. \end{aligned} \tag{$*$}$$

Clearly, $|B(f)|$ is always even, and if $B(f) = \emptyset$, then either $C(f) = \emptyset$ or $C(f) = V(f)$. Also note that f sends nothing through a vertex u of f if f has a vertex v such that uv is a common edge of f and a 3-face of G.

Suppose $|f| = 9$. By inequality $(*)$, it suffices to consider following three cases.

Case 1: $|A(f)| \leq 2$ and $|B(f)| = |C(f)| = 0$. By Lemma 2.9, we have $|A(f)| = 2$ (say $A(f) = \{u, v\}$), $D(f)$ is divided by u and v as 3+4 on f, and $d(u), d(v) \geq 4$. Through the drawing of 3-faces adjacent to f, one can find that Lemma 2.13 implies that not both u and v have degree 4. Hence, we have $ch^*(f) \geq |f| - 4 - 7 \times \frac{2}{3} - \frac{1}{3} = 0$.

Case 2: $|A(f)| = 1, |B(f)| = 2$ and $|C(f)| = 0$. By Lemma 2.9, $D(f)$ is divided by $B(f) \cup A(f)$ as 3+3 or 2+4 on f.

In the former case 3+3, let $A(f) = \{u\}$. By Lemma 2.13, u is not a 4-vertex incident with two 3-faces, and thus receives at most $\frac{1}{6}$ from f. Hence, we have $ch^*(f) \geq |f| - 4 - 6 \times \frac{2}{3} - 2 \times \frac{3}{8} - \frac{1}{6} > 0$.

In the latter case 2+4, let $f = [u_1 \ldots u_9]$, $u_1 \in A(f)$, and $u_4, u_5 \in B(f)$. Lemma 2.9 implies $d(u_1), d(u_5) \geq 4$. Furthermore, u_1 is a 4-vertex incident with two 3-faces, since otherwise f sends at most $\frac{1}{6}$ to u_1 so that $ch^*(f) \geq |f| - 4 - 6 \times \frac{2}{3} - 2 \times \frac{3}{8} - \frac{1}{6} > 0$. Through the drawing of 3-faces adjacent to f, one can find that $d(u_4) \geq 4$. Hence, f sends nothing through u_4 and u_5, and at most $\frac{1}{3}$ to each of them, yielding $ch^*(f) \geq |f| - 4 - 6 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$.

Case 3: $|A(f)| = 0, |B(f)| = 2$ and $|C(f)| \leq 2$. It follows that f contains five consecutive bad vertices, and hence has a good path, contradicting Lemma 2.9.

Suppose $|f| \geq 10$. If $|A(f)| + \frac{|B(f)|}{2} \geq 2$, then by inequality $(*)$ we are done. Hence, we may assume either $|A(f)| \leq 1$ and $|B(f)| = 0$, or $|A(f)| = 0$ and $|B(f)| = 2$. Lemma 2.9 implies a contradiction in the former case, and $|C(f)| \geq 4$ in the latter case. Hence, by inequality $(*)$ we are also done in the latter case.

Next suppose f contains external vertices. Since $|f_0| \leq 12$, if $f = f_0$ then by $R5$ we have $ch^*(f) = |f_0| + 4 - |f_0| \times \frac{4}{3} \geq 0$. Hence, we may assume $f \neq f_0$. By our rules, f sends at most $\frac{2}{3}$ to each incident vertex. Lemma 2.7 implies that if $|f| \leq 8$, then the external vertices on f are consecutive one by one. Furthermore, f has at most one 2-vertex if $|f| = 5$, and has at most two 2-vertices if $|f| \in \{7, 8\}$.

Let $|f| = 3$. We have $ch^*(f) = |f| - 4 + 3 \times \frac{1}{3} = 0$ by $R1$.

Let $|f| = 5$. If f has no 2-vertex, then f sends at most $\frac{1}{4}$ to each vertex, yielding $ch^*(f) \geq |f| - 4 - 4 \times \frac{1}{4} = 0$. Hence, we may assume f has precisely one 2-vertex. It follows that f has two external 3-vertices, both of which send at least $\frac{1}{12}$ to f by $R6$. Hence, we have $ch^*(f) \geq |f| - 4 - \frac{2}{3} + 2 \times \frac{1}{12} - 2 \times \frac{1}{4} = 0$.

Let $|f| = 7$. Since in this case f is adjacent to at most one 3-face, f has an internal vertex that is not bad. By our rules, f sends at most $\frac{1}{2}$ to this vertex. If f has an external 4^+-vertex, then f receives $\frac{1}{3}$ from this vertex by $R6(3)$, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{3} - 4 \times \frac{2}{3} - \frac{1}{2} > 0$. Hence, we may assume that f has no external 4^+-vertex, which implies f has two external 3-vertices u and v. If both of u and v are not triangular and thus send $\frac{1}{12}$ to f, then we have $ch^*(f) \geq |f| - 4 + 2 \times \frac{1}{12} - 4 \times \frac{2}{3} - \frac{1}{2} = 0$. Hence, we may assume that u is triangular but v not. Now f has at most one bad vertex, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{12} - \frac{1}{12} - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$.

Let $|f| = 8$. If f has no 2-vertex, then f sends at most $\frac{1}{2}$ to each incident vertex, yielding $ch^*(f) \geq |f| - 4 - 8 \times \frac{1}{2} = 0$. Hence, we may assume that f has precisely one or two 2-vertices. It follows that f has two external 3^+-vertices, both of which send at least $\frac{1}{12}$ to f. Hence we have $ch^*(f) \geq |f| - 4 - 2 \times \frac{2}{3} + 2 \times \frac{1}{12} - 4 \times \frac{1}{2} > 0$.

It remains to suppose $|f| \geq 9$. If f has an external 4^+-vertex, then f receives $\frac{1}{3}$ from this vertex by $R6(3)$, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{3} - (|f| - 1) \times \frac{2}{3} \geq 0$. Hence, we may assume that f has no external 4^+-vertex, which implies f has at least two external 3-vertices. By $R6$, we have $ch^*(f) \geq |f| - 4 - 2 \times \frac{1}{12} - (|f| - 2) \times \frac{2}{3} > 0$.

Claim 2.15. *$ch^*(v) \geq 0$ for $v \in V$.*

First suppose that v is internal. We have $d(v) \geq 3$ by Lemma 2.1.

Let $d(v) = 3$. Since $G \in \mathcal{G}$, the set of lengths of the faces containing v is one of the followings: $\{3, 7^+, 7^+\}, \{5, 5, 7^+\}, \{5, 7^+, 7^+\}$ and $\{7^+, 7^+, 7^+\}$. Hence, we are done in each case by $R1$ and $R2$.

If $d(v) = 4$, then by $R1$ and $R3$ the charge v sends out equals to the charge v receives, yielding that $ch^*(v) = d(v) - 4 = 0$.

It remains to suppose $d(v) \geq 5$. By $R1$ and $R4(1)$, v sends $\frac{1}{3}$ to each incident 3-face and at most $\frac{1}{24}$ to each other incident face, which gives $ch^*(v) > d(v) - 4 - \frac{d(v)}{2} \times \frac{1}{3} - \frac{d(v)}{2} \times \frac{1}{24} > 0$.

Next suppose that v is external. Clearly, $d(v) \geq 2$.

By $R1$, $R5$ and $R6$, we have $ch^*(v) = d(v) - 4 + \frac{4}{3} + \frac{2}{3} = 0$ if $d(v) = 2$, $ch^*(v) = d(v) - 4 + \frac{4}{3} - \frac{1}{3} + \frac{1}{12} > 0$ if $d(v) = 3$ and v is triangular, and $ch^*(v) = d(v) - 4 + \frac{4}{3} - \frac{1}{12} - \frac{1}{12} > 0$ if $d(v) = 3$ and v is not triangular.

It remains to suppose $d(v) \geq 4$. Then v receives $\frac{4}{3}$ from f_0 by $R5$, sends $\frac{1}{3}$ to each other incident face by $R1$ and $R6(3)$, and might sends $\frac{5}{24}$ through each incident 3-face whose other two vertices are internal. It follows that $ch^*(v) \geq d(v) - 4 + \frac{4}{3} - (d(v) - 1) \times \frac{1}{3} - \frac{d(v)-2}{2} \times \frac{5}{24} > 0$.

Claim 2.16. *D contains a vertex x_0 such that $ch^*(x_0) > 0$.*

Let x_0 be any 3^+-vertex on D, as desired.

The proof of Theorem 1.5 is completed.

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